GENERAL CHARACTERIZATION THEOREMS AND INTRINSIC TOPOLOGIES IN WHITE NOISE ANALYSIS

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ABSTRACT. Let u be a positive continuous function on $[0,\infty)$ satisfying the conditions: (i) $\lim_{r\to\infty} r^{-1/2}\log u(r)=\infty$, (ii) $\inf_{r\ge 0} u(r)=1$, (iii) $\lim_{r\to\infty} r^{-1}\log u(r)<\infty$, (iv) the function $\log u(x^2)$, $x\ge 0$, is convex. A Gel'fand triple $[\mathcal{E}]_u\subset (L^2)\subset [\mathcal{E}]_u^*$ is constructed by making use of the Legendre transform of u discussed in [4]. We prove a characterization theorem for generalized functions in $[\mathcal{E}]_u^*$ and also for test functions in $[\mathcal{E}]_u$ in terms of their S-transforms under the same assumptions on u. Moreover, we give an intrinsic topology for the space $[\mathcal{E}]_u$ of test functions and prove a characterization theorem for measures. We briefly mention the relationship between our method and a recent work by Gannoun et al.[10]. Finally, conditions for carrying out white noise operator theory and Wick products are given.

1. Introduction

Let $\mathcal{E}$ be a real topological vector space with topology generated by a sequence of inner product norms $\{|\cdot|_p\}_{p=0}^\infty$. We assume that $\mathcal{E}$ is a complete metric space with respect to the metric

$$d(\xi,\eta)=\sum_{p=0}^\infty \frac{1}{2^p}\frac{|\xi-\eta|_p}{1+|\xi-\eta|_p},\qquad \xi,\eta\in\mathcal{E}.$$

In addition we assume the following conditions:

(a) There exists a constant $0<\rho<1$ such that $|\cdot|_0\le\rho|\cdot|_1\le\cdots\le\rho^p|\cdot|_p\le\cdots$.
(b) For any $p\ge 0$, there exists $q\ge p$ such that the inclusion $i_{q,p}:\mathcal{E}_q\hookrightarrow\mathcal{E}_p$ is a Hilbert-Schmidt operator. (Here $\mathcal{E}_p$ is the completion of $\mathcal{E}$ with respect to the norm $|\cdot|_p$.)

Let $\mathcal{E}'$ and $\mathcal{E}_p'$ denote the dual spaces of $\mathcal{E}$ and $\mathcal{E}_p$, respectively. We can use the Riesz representation theorem to identify $\mathcal{E}_0$ with its dual space $\mathcal{E}_0'$. Then we get the following continuous inclusions:

$$\mathcal{E}\hookrightarrow\mathcal{E}_p\hookrightarrow\mathcal{E}_0\hookrightarrow\mathcal{E}_p'\hookrightarrow\mathcal{E}',\qquad p\ge 0.$$

The above condition (b) says that $\mathcal{E}$ is a nuclear space and so $\mathcal{E}\subset\mathcal{E}_0\subset\mathcal{E}'$ is a Gel'fand triple. Let μ be the Gaussian measure on $\mathcal{E}'$ with the characteristic function given by

$$\int_{\mathcal{E}'} e^{i\langle x,\xi\rangle}\,d\mu(x)=e^{-\frac12|\xi|_0^2},\qquad \xi\in\mathcal{E}.$$

The probability space $(\mathcal{E}',\mu)$ is often referred to as a *white noise space*. For simplicity, we will use (L^2) to denote the complex Hilbert space $L^2(\mu)$. By the

Wiener-Itô theorem, each $\varphi \in (L^2)$ can be uniquely represented by

$$\varphi(x) = \sum_{n=0}^{\infty} I_n(f_n)(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f_n \rangle, \qquad f_n \in \mathcal{E}_0^{\widehat{\otimes} n}, \tag{1.1}$$

where I_n is the multiple Wiener integral of order n and $:x^{\otimes n}:$ is the Wick tensor of $x \in \mathcal{E}'$ (see page 33 in [21].) Moreover, the (L^2)-norm of φ is given by

$$\|\varphi\|_0 = \left(\sum_{n=0}^{\infty} n! |f_n|_0^2 \right)^{1/2}. \tag{1.2}$$

Recently Cochran et al. [8] have introduced a Gel'fand triple associated with the above Gel'fand triple $\mathcal{E} \subset \mathcal{E}_0 \subset \mathcal{E}'$ and a sequence $\{\alpha(n)\}_{n=0}^{\infty}$ of positive real numbers satisfying the conditions:

(A1) $\alpha(0) = 1$ and $\inf_{n \geq 0} \alpha(n)\sigma^n > 0$ for some $\sigma \geq 1$.

(A2) $\lim_{n\to\infty} \left(\frac{\alpha(n)}{n!} \right)^{1/n} = 0$.

Actually, a stronger condition $\inf_{n\geq 0} \alpha(n) > 0$ in (A1) is assumed in [8]. However the above weaker condition for some $\sigma \geq 1$ in (A1) is strong enough to assure that the nuclear space $[\mathcal{E}]_\alpha$ is a subspace of (L^2), a fact to be shown below. This weaker condition was first introduced in [5].

For $\varphi \in (L^2)$ being represented by Equation (1.1) and $p \geq 0$, define

$$\|\varphi\|_{p,\alpha} = \left(\sum_{n=0}^{\infty} n!\alpha(n) |f_n|_p^2 \right)^{1/2}. \tag{1.3}$$

Let $[\mathcal{E}_p]_\alpha = \{\varphi \in (L^2);\ \|\varphi\|_{p,\alpha} < \infty\}$. Define the space $[\mathcal{E}]_\alpha$ of *test functions* on $\mathcal{E}'$ to be the projective limit of $\{[\mathcal{E}_p]_\alpha;\ p \geq 0\}$. The dual space $[\mathcal{E}]_\alpha^*$ of $[\mathcal{E}]_\alpha$ is called the space of *generalized functions* on $\mathcal{E}'$.

By using conditions (a) and (A1) we see that

$$\sum_{n=0}^{\infty} n!\alpha(n)|f_n|_p^2 \geq \left(\inf_{n\geq 0} \alpha(n)\sigma^n \right) \sum_{n=0}^{\infty} n!|f_n|_0^2$$

for p large enough such that $\sigma^{-1}\rho^{-2p} \geq 1$. This inequality, in view of Equations (1.2) and (1.3), implies that $[\mathcal{E}_p]_\alpha \subset (L^2)$ for all $p \geq (-2\log\rho)^{-1}\log\sigma$. Hence $[\mathcal{E}]_\alpha \subset (L^2)$ holds. By identifying (L^2) with its dual space we get the following continuous inclusions:

$$[\mathcal{E}]_\alpha \hookrightarrow [\mathcal{E}_p]_\alpha \hookrightarrow (L^2) \hookrightarrow [\mathcal{E}_p]_\alpha^* \hookrightarrow [\mathcal{E}]_\alpha^*, \qquad p \geq (-2\log\rho)^{-1}\log\sigma,$$

where $[\mathcal{E}_p]_\alpha^*$ is the dual space of $[\mathcal{E}_p]_\alpha$. Moreover, $[\mathcal{E}]_\alpha$ is a nuclear space and so $[\mathcal{E}]_\alpha \subset (L^2) \subset [\mathcal{E}]_\alpha^*$ is a Gel'fand triple. This triple is refered to as a *CKS-space.* Note that $[\mathcal{E}]_\alpha^* = \cup_{p\geq 0}[\mathcal{E}_p]_\alpha^*$ and for $p \geq (-2\log\rho)^{-1}\log\sigma$, $[\mathcal{E}_p]_\alpha^*$ is the completion of (L^2) with respect to the norm

$$\|\varphi\|_{-p,1/\alpha} = \left(\sum_{n=0}^{\infty} \frac{n!}{\alpha(n)} |f_n|_{-p}^2 \right)^{1/2}. \tag{1.4}$$

For ξ belonging to the complexification $\mathcal{E}_c$ of $\mathcal{E}$, the renormalized exponential function $:e^{\langle\cdot,\xi\rangle}:$ is defined by

$$:e^{\langle\cdot,\xi\rangle}: = \sum_{n=0}^{\infty} \frac{1}{n!} \langle :\cdot^{\otimes n}:, \xi^{\otimes n}\rangle = e^{\langle\cdot,\xi\rangle - \frac{1}{2}\langle\xi,\xi\rangle},$$

whose norm is evaluated as

$$\| :e^{\langle\cdot,\xi\rangle}: \|_{p,\alpha} = G_\alpha\big(|\xi|_p^2\big)^{1/2}, \quad p \geq 0, \tag{1.5}$$

where G_α is the exponential generating function of the sequence $\{\alpha(n)\}$, i.e.,

$$G_\alpha(r) = \sum_{n=0}^{\infty} \frac{\alpha(n)}{n!} r^n. \tag{1.6}$$

By condition (A2) G_α is an entire function. Hence Equation (1.5) implies that $:e^{\langle\cdot,\xi\rangle}: \in [\mathcal{E}]_\alpha$ for all $\xi \in \mathcal{E}_c$.

On the other hand, by Equation (1.4), we have

$$\| :e^{\langle\cdot,\xi\rangle}: \|_{-p,1/\alpha} = G_{1/\alpha}\big(|\xi|_{-p}^2\big)^{1/2}, \tag{1.7}$$

where $G_{1/\alpha}$ is the exponential generating function of the sequence $\{1/\alpha(n)\}$, i.e.,

$$G_{1/\alpha}(r) = \sum_{n=0}^{\infty} \frac{1}{n!\alpha(n)} r^n. \tag{1.8}$$

It follows from condition (A1) that $G_{1/\alpha}$ is an entire function.

For $\Phi \in [\mathcal{E}]_\alpha^*$, its *S-transform* $S\Phi$ is defined to be the function

$$(S\Phi)(\xi) = \langle\!\langle \Phi, :e^{\langle\cdot,\xi\rangle}: \rangle\!\rangle, \qquad \xi \in \mathcal{E}_c,$$

where $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ is the bilinear pairing of $[\mathcal{E}]_\alpha^*$ and $[\mathcal{E}]_\alpha$.

An important problem in white noise analysis is to characterize generalized and test functions in terms of their S-transforms. For this purpose we need the following conditions:

(B1) $\limsup_{n\to\infty} \left(\frac{n!}{\alpha(n)} \inf_{r>0} \frac{G_\alpha(r)}{r^n}\right)^{1/n} < \infty$.

($\widetilde{\mathrm{B}}1$) $\limsup_{n\to\infty} \left(n!\alpha(n) \inf_{r>0} \frac{G_{1/\alpha}(r)}{r^n}\right)^{1/n} < \infty$.

(B2) The sequence $\gamma(n) = \frac{\alpha(n)}{n!}, n \geq 0$, is log-concave, i.e.,

$$\gamma(n)\gamma(n+2) \leq \gamma(n+1)^2, \qquad \forall n \geq 0.$$

($\widetilde{\mathrm{B}}2$) The sequence $\left\{\frac{1}{n!\alpha(n)}\right\}$ is log-concave.

It follows from Theorem 4.3 in [8] that condition (B2) implies condition (B1). Similarly, condition ($\widetilde{\mathrm{B}}2$) implies condition ($\widetilde{\mathrm{B}}1$), [3]. Characterization thorems are proved in [8] for generalized functions under (B2) and in [3] for test functions under ($\widetilde{\mathrm{B}}2$). In fact, those conditions can be replaced by weaker conditions. We say that two sequences $\{a(n)\}$ and $\{b(n)\}$ of positive real numbers are *equivalent* if there exist $K_1, K_2, c_1, c_2 > 0$ such that for all n,

$$K_1 c_1^n a(n) \leq b(n) \leq K_2 c_2^n a(n).$$

Now, we state the weaker conditions for the sequence $\{\alpha(n)\}$:

Near-(B2) The sequence $\{\alpha(n)\}$ is equivalent to a sequence $\{\lambda(n)\}$ of positive real numbers such that $\{\lambda(n)/n!\}$ is log-concave.

Near-($\widetilde{\text{B}}$2) The sequence $\{\alpha(n)\}$ is equivalent to a sequence $\{\lambda(n)\}$ of positive real numbers such that $\{\frac{1}{n!\lambda(n)}\}$ is log-concave.

As shown in Lemma 2.1 later, near-(B2) and near-($\widetilde{\text{B}}$2) are equivalent to the conditions (B1) and ($\widetilde{\text{B}}$1), respectively. Then, we have the following theorems.

Theorem 1.1. *If $F = S\Phi$ for $\Phi \in [\mathcal{E}]^*_\alpha$, then F satisfies the conditions:*

(1) *For any $\xi, \eta \in \mathcal{E}_c$, the function $F(z\xi + \eta)$ is an entire function of $z \in \mathbb{C}$.*
(2) *There exist constants $K, a, p \geq 0$ such that*

$$|F(\xi)| \leq K G_\alpha\big(a|\xi|_p^2\big)^{1/2}, \qquad \xi \in \mathcal{E}_c.$$

Conversely, assume that condition near-(B2) holds and let $F\colon \mathcal{E}_c \to \mathbb{C}$ be a function satisfying conditions (1) and (2). Then $F = S\Phi$ for a unique generalized function $\Phi \in [\mathcal{E}]^*_\alpha$.

Theorem 1.2. *If $F = S\varphi$ for $\varphi \in [\mathcal{E}]_\alpha$, then F satisfies the conditions:*

(1) *For any $\xi, \eta \in \mathcal{E}_c$, the function $F(z\xi + \eta)$ is an entire function of $z \in \mathbb{C}$.*
(2) *For any $a, p \geq 0$, there exists a constant $K \geq 0$ such that*

$$|F(\xi)| \leq K G_{1/\alpha}\big(a|\xi|_{-p}^2\big)^{1/2}, \qquad \xi \in \mathcal{E}_c.$$

Conversely, assume that condition near-($\widetilde{B}$2) holds and let $F\colon \mathcal{E}_c \to \mathbb{C}$ be a function satisfying conditions (1) and (2). Then $F = S\varphi$ for a unique test function $\varphi \in [\mathcal{E}]_\alpha$.

Now, for a general sequence $\{\alpha(n)\}$, we cannot expect to find the sums G_α in Equation (1.6) and $G_{1/\alpha}$ in Equation (1.8) as elementary functions. Therefore, it is desirable to find elementary functions to replace G_α and $G_{1/\alpha}$ in Theorems 1.1 and 1.2. This leads to the concept of equivalence in the next definition.

Definition 1.3. Two positive functions f and g on $[0, \infty)$ are called *equivalent* if there exist constants $c_1, c_2, a_1, a_2 > 0$ such that

$$c_1 f(a_1 r) \leq g(r) \leq c_2 f(a_2 r), \qquad \forall\, r \in [0, \infty).$$

In order to find elementary functions that are equivalent to G_α and $G_{1/\alpha}$ in Theorems 1.1 and 1.2, we have developed in [4] the crucial mathematical machinery.

Example 1.1. When $\alpha(n) = 1$ for all n, condition (B2) is obviously satisfied and $G_\alpha(r) = G_{1/\alpha}(r) = e^r$. In this case defined by Hida-Kubo-Takenaka, Theorem 1.1 is due to Potthoff and Streit [29], while Theorem 1.2 is due to Kuo et al. [24].

Example 1.2. When $\alpha(n) = (n!)^\beta$, $0 \leq \beta < 1$, condition (B2) is easily seen to be satisfied. The functions G_α and $G_{1/\alpha}$ are given respectively by

$$G^{(\beta)}(r) = \sum_{n=0}^\infty \frac{1}{(n!)^{1-\beta}}\, r^n, \quad G^{(-\beta)}(r) = \sum_{n=0}^\infty \frac{1}{(n!)^{1+\beta}}\, r^n. \tag{1.9}$$

However we see that $G^{(\beta)}$ and $G^{(-\beta)}$ are equivalent to the functions

$$g_\beta(r) = \exp\Big[(1-\beta) r^{\frac{1}{1-\beta}}\Big], \quad g_{-\beta}(r) = \exp\Big[(1+\beta) r^{\frac{1}{1+\beta}}\Big], \tag{1.10}$$

respectively. Theorems 1.1 and 1.2 with the growth functions g_β and $g_{-\beta}$, respectively, are due to Kondratiev and Streit [12] [13] (see also [21].)

Example 1.3. When $\alpha(n) = b_k(n)$ (the Bell numbers of order k,) condition (B1) is shown to be satisfied in [8]. However actually condition (B2) is satisfied [2]. In this case, $G_\alpha(r) = \exp_k(x)/\exp_k(0)$ ($\exp_k(x)$ is the k-th iterated exponential function) and Theorem 1.1 is due to Cochran et al. [8]. However, $G_{1/\alpha}$ is not an elementary function. However it is equivalent to the function

$$w_k(r) = \exp\left[2\sqrt{r\log_{k-1}\sqrt{r}}\right], \tag{1.11}$$

where the function $\log_j$ is defined inductively by

$$\log_1(r) = \log(\max\{r, e\}), \quad \log_j(r) = \log_1(\log_{j-1}(r)), \quad j \geq 2.$$

The first purpose of the present paper is to construct a CKS-space $[\mathcal{E}]_u \subset (L^2) \subset [\mathcal{E}]_u^*$ with a given growth function u and to obtain the general characterization theorems by applying the results in [4]. The second purpose is to give the intrinsic topology for $[\mathcal{E}]_u$ and to show properties of the space relating to the features of u. The basic idea is to start with a growth function u and then apply the Legendre transform to get a Gel'fand triple $[\mathcal{E}]_u \subset (L^2) \subset [\mathcal{E}]_u^*$.

We remark that Gannoun et al. [10] studied a similar Gel'fand triple consisting of spaces of entire functions governed by a convex function θ and its dual. Their inclusions and duality are rather abstract. We will give comments about relationships between u and θ in section 4.

Further we will discuss the characterization of measures in $[\mathcal{E}]_u^*$ by an integrability condition in section 4. Ouerdiane and Rezgui [28] showed the Bochner-Minlos theorem, which tells an integrability condition of measures in terms of growth order of characteristic functions.

2. Legendre and dual Legendre transforms

First we mention the following concepts which will be frequently used. A positive function f on $[0, \infty)$ is called

(1) *log-concave* if the function $\log f$ is concave on $[0, \infty)$;
(2) *log-convex* if the function $\log f$ is convex on $[0, \infty)$;
(3) *(log, exp)-convex* if the function $\log f(e^x)$ is convex on $\mathbb{R}$;
(4) *(log, x^k)-convex* if the function $\log f(x^k)$ is convex on $[0, \infty)$. Here $k > 0$.

It is easy to check that if f is log-concave, then the sequence $\{f(n)\}_{n=0}^\infty$ is log-concave. If f is increasing and (log, x^k)-convex for some $k > 0$, then f is (log, exp)-convex (see Proposition 2.3(3) in [3].) Further, if $\{\beta(n)/n!\}_{n=0}^\infty$ is log-concave and $\beta(0) = 1$, then for any $n, m \geq 0$,

$$\beta(n+m) \leq \binom{n+m}{n}\beta(n)\beta(m) \leq 2^{n+m}\beta(n)\beta(m). \tag{2.1}$$

Let $C_{+,\log}$ denote the collection of all positive continuous functions u on $[0, \infty)$ satisfying the condition:

$$\lim_{r\to\infty} \frac{\log u(r)}{\log r} = \infty.$$

The *Legendre transform* ℓ_u of $u \in C_{+,\log}$ is defined to be the function

$$\ell_u(t) = \inf_{r>0} \frac{u(r)}{r^t}, \qquad t \in [0, \infty).$$

Let $C_{+,\,1/2}$ denote the collection of all positive continuous functions u on $[0, \infty)$ satisfying the condition:

$$\lim_{r\to\infty} \frac{\log u(r)}{\sqrt{r}} = \infty.$$

The *dual Legendre transform* u^* of $u \in C_{+,1/2}$ is defined to be the function

$$u^*(r) = \sup_{s\geq 0} \frac{e^{2\sqrt{rs}}}{u(s)}, \qquad r \in [0, \infty).$$

Note that $C_{+,1/2} \subset C_{+,\log}$. Assume that $u \in C_{+,\log}$ and $\lim_{n\to\infty} \ell_u(n)^{1/n} = 0$. We define the *L-function* $\mathcal{L}_u$ of u by

$$\mathcal{L}_u(r) = \sum_{n=0}^{\infty} \ell_u(n) r^n. \tag{2.2}$$

Now, let $u \in C_{+,1/2}$ and assume that $\lim_{n\to\infty} \big(\ell_u(n)(n!)^2\big)^{-1/n} = 0$. We define the *$L^{\#}$-function* of u by

$$\mathcal{L}_u^{\#}(r) = \sum_{n=0}^{\infty} \frac{1}{\ell_u(n)(n!)^2}\, r^n. \tag{2.3}$$

For discussions in the rest of the paper, we will need the following facts from papers [4] [5].

Fact 2.1. *The inclusion $C_{+,\frac{1}{2}} \subset C_{+,\log}$ holds. If u is increasing and $(\log, x^2)$-convex, then u is (log, exp)-convex.*

Fact 2.2. *Let $u \in C_{+,\log}$. Then the Legendre transform ℓ_u is log-concave. (Hence ℓ_u is continuous on $[0, \infty)$ and the sequence $\{\ell_u(n)\}_{n=0}^{\infty}$ is log-concave.)*

Fact 2.3. *Let $u \in C_{+,\log}$ be (log, exp)-convex. Then*

(1) *$\ell_u(t)$ is decreasing for large t,*
(2) $\lim_{t\to\infty} \ell_u(t)^{1/t} = 0$,
(3) *$u(r) = \sup_{t\geq 0} \ell_u(t) r^t$ for all $r \geq 0$.*

Fact 2.4. *Let $u \in C_{+,\log}$. We have the assertions:*

(1) *u is (log, x^k)-convex if and only if $\ell_u(t)t^{kt}$ is log-convex.*
(2) *If u is (log, x^k)-convex, then for any integers $n, m \geq 0$,*

$$\ell_u(n)\ell_u(m) \leq \ell_u(0) 2^{k(n+m)} \ell_u(n + m).$$

Fact 2.5. *(1) Let $u \in C_{+,\log}$ be (log, exp)-convex. Then its L-function $\mathcal{L}_u$ is also (log, exp)-convex and for any $a > 1$,*

$$\mathcal{L}_u(r) \leq \frac{ea}{\log a} u(ar), \qquad \forall r \geq 0.$$

(2) Let $u \in C_{+,\log}$ be increasing and (log, x^k)-convex. Then there exists a constant C, independent of k, such that

$$u(r) \leq C\mathcal{L}_u(2^k r), \qquad \forall r \geq 0.$$

Fact 2.6. *Let $u \in C_{+,1/2}$. Then its dual Legendre transform u^* belongs to $C_{+,1/2}$ and is an increasing (log, x^2)-convex function on $[0, \infty)$.*

Fact 2.7. *If $u \in C_{+,1/2}$ is (log, x^2)-convex, then the Legendre transform ℓ_{u^*} of u^* is given by*

$$\ell_{u^*}(t) = \frac{e^{2t}}{\ell_u(t)t^{2t}}, \qquad t \in [0, \infty).$$

Fact 2.8. *Let $u \in C_{+,1/2}$ be (log, x^2)-convex. If u is increasing on the interval $[r_0, \infty)$, then we have $(u^*)^*(r) = u(r)$ for all $r \geq r_0$. In particular, if u is increasing on $[0, \infty)$, then $(u^*)^* = u$ on $[0, \infty)$.*

Fact 2.9. *Let $u \in C_{+,1/2}$ be (log, x^2)-convex. Then the functions u^*, $\mathcal{L}_{u^*}$, and $\mathcal{L}_u^{\#}$ are all equivalent.*

Lemma 2.1. *The conditions (B1) and ($\widetilde{B}1$) are equivalent to near-(B2) and near-($\widetilde{B}2$), respectively.*

Proof. It is enough to show the equivalence of (B1) and near-(B2). Put $u(r) = G_\alpha(r)$. It is easy to see that $u \in C_{+,\log}$ and $\alpha(n)/n! \leq \ell_u(n)$. By Fact 2.2, $\ell_u(n)$ is log-concave. Since $\inf_{r>0} G_\alpha(r)/r^n = \ell_u(n)$, the condition (B1) is equivalent to that there exists a positive constant C such that $\ell_u(n) \leq C^n \alpha(n)/n!$. Hence $\{\alpha(n)/n!\}$ is equivalent to the log-concave sequence $\{\ell_u(n)\}$, if (B1) holds.

Conversely, suppose that there exists a positive sequence $\{\beta(n)\}$ and positive constants K_1, K_2, c_1, c_2 such that $\{\beta(n)/n!\}$ is log-concave.

$$K_1 c_1^n \frac{\beta(n)}{n!} \leq \frac{\alpha(n)}{n!} \leq K_2 c_2^n \frac{\beta(n)}{n!}.$$

Then we have

$$K_1 G_\beta(c_1 r) \leq G_\alpha(r) \leq K_2 G_\beta(c_2 r).$$

Therefore, we obtain

$$\begin{aligned}\limsup_{n\to\infty} \left(\frac{n!}{\alpha(n)} \inf_{r>0} \frac{G_\alpha(r)}{r^n} \right)^{1/n} &\leq \limsup_{n\to\infty} \left(\frac{n!}{K_1 c_1^n \beta(n)} \inf_{r>0} \frac{K_2 G_\beta(c_2 r)}{r^n} \right)^{1/n} \\ &= \frac{K_2 c_2}{K_1 c_1} \limsup_{n\to\infty} \left(\frac{n!}{\beta(n)} \inf_{r>0} \frac{G_\beta(r)}{r^n} \right)^{1/n} < \infty,\end{aligned}$$

since the condition (B2) for $\{\beta(n)\}$ implies (B1) for $\{\beta(n)\}$ (see [8]). □

For more precise discussion, We will need the following conditions on $u \in C_{+,1/2}$:

(U0) $\inf_{r\geq 0} u(r) = 1$.

(U1) u is increasing and $u(0) = 1$.

(U2) $\lim_{r\to\infty} r^{-1}\log u(r) < \infty$.

(U3) u is (log, x^2)-convex.

Recall that the A-conditions are needed in order to set up the Gel'fand triple $[\mathcal{E}]_\alpha \subset (L^2) \subset [\mathcal{E}]_\alpha^*$ and to make sure that the renormalized exponential functions $:e^{\langle \cdot,\xi\rangle}:$, $\xi \in \mathcal{E}_c$, are test functions in $[\mathcal{E}]_\alpha$. Moreover, note that the B-conditions are used for the characterization theorems [3] [8]. Keeping these in mind, we consider the relationship between the U-conditions and AB-conditions in the rest of this section.

For a given $u \in C_{+,\log}$, define a sequence $\{\alpha_u(n)\}$ by

$$\alpha_u(n) = \frac{1}{n!\ell_u(n)}. \tag{2.4}$$

Lemma 2.2. *If $u \in C_{+,\log}$ satisfies conditions $(U0)$ and $(U2)$, then α_u satisfies the condition (A1).*

Proof. By the definition of Legendre transform, $\ell_u(0) = 1$. Since u satisfies condition (U2), there exist constants $c_1, c_2 > 0$ such that $u(r) \leq c_1 e^{c_2 r}$ for all $r \geq 0$. Therefore,

$$\ell_u(n) = \inf_{r>0} \frac{u(r)}{r^n} \leq \inf_{r>0} \frac{c_1 e^{c_2 r}}{r^n} = c_1 c_2^n \left(\frac{e}{n}\right)^n \leq c_1 e \frac{(c_2\sqrt{2}\,)^n}{n!}.$$

by the Stirling formula $n! \leq e\sqrt{n}\big(n/e\big)^n$. Therefore $\alpha_u(n)(c_2\sqrt{2}\,) \geq (c_1 e)^{-1}$. $\square$

Further, we can show the condition (A2) by the following lemma.

Lemma 2.3. *Let $u \in C_{+,1/2}$ satisfy condition (U3) and α_u be in (2.4). Then, α_u satisfies the condition (A2). In addition, G_{α_u} defined in Equations (1.8) and $\mathcal{L}_u^{\#}$ in (2.3) are the same entire function, i.e. $G_{\alpha_u}(r) = \mathcal{L}_u^{\#}(r)$.*

Proof. The equality is obvious. By the condition $u \in C_{+,1/2}$ and by Fact 2.6 the dual transform u^* belongs to $C_{+,1/2}$ and is an increasing (log, x^2)-convex function. By Fact 2.1, u^* belongs to $C_{+,\log}$ and is (log, exp)-convex. Therefore ℓ_{u^*} satisfies (2) in Fact 2.3. Then we see

$$\lim_{n\to\infty} \left(\frac{1}{\ell_u(n)(n!)^2}\right)^{1/n} = \lim_{n\to\infty} \left(\frac{\ell_{u^*}(n)n^{2n}}{(n!)^2 e^{2n}}\right)^{1/n} = \lim_{n\to\infty} \ell_{u^*}(n)^{1/n} = 0$$

by Fact 2.7 and the Stirling formula we have the condition (A2). $\square$

Lemma 2.4. *If $u \in C_{+,\log}$ satisfies condition (U3), then the sequence $\{\alpha_u(n)\}$ satisfies condition near-(B2).*

Remark. Even if we assume that $u \in C_{+,1/2}$ with condition (U3), we cannot conclude that $\{\alpha_u(n)\}$ satisfies condition (B2). On the other hand, if we assume that $u \in C_{+,\log}$ is log-convex, then $\{\alpha_u(n)\}$ does satisfy condition (B2). For the proof, see Lemma 3.3 in [5].

Proof. We can apply Fact 2.4 (1) to see that $\{\ell_u(n)n^{2n}\}$ is log-convex. However $\ell_u(n) = (\alpha_u(n)n!)^{-1}$. Hence the sequence $\{(\alpha_u(n)n!)^{-1}n^{2n}\}$ is log-convex and so the sequence $\{\alpha_u(n)n!/n^{2n}\}$ is log-concave.

Let $\lambda(n) = \alpha_u(n)(n!)^2/n^{2n}$. We have just shown that $\{\lambda(n)/n!\}$ is log-concave. On the other hand, it follows from the Stirling formula that $\{\alpha_u(n)\}$ and $\{\lambda(n)\}$ are equivalent. Hence $\{\alpha(n)\}$ satisfies condition near-(B2). $\square$

Lemma 2.5. *Let $u \in C_{+,\log}$. Then the sequence $\{\alpha_u(n)\}$ satisfies condition $(\widetilde{B}2)$.*

Proof. By Fact 2.2, $\{\ell_u(n)\}$ is log-concave. However $\ell_u(n) = (n!\alpha_u(n))^{-1}$ and so the sequence $\{(n!\alpha_u(n))^{-1}\}$ is log-concave. This means that the sequence $\{\alpha_u(n)\}$ satisfies condition $(\widetilde{\mathrm{B}}2)$. $\square$

Putting the above four lemmas together, we get the next theorem for the Gel'fand triple $[\mathcal{E}]_u \subset (L^2) \subset [\mathcal{E}]_u^*$ associated with a growth function u.

Theorem 2.6. *Suppose $u \in C_{+,1/2}$ satisfies conditions (U0) (U2) (U3). Then the sequence $\alpha_u(n) = \big(\ell_u(n)n!\big)^{-1}$, $n \geq 0$, satisfies conditions (A1), (A2), near-(B2), and $(\widetilde{B}2)$.*

3. Characterization theorems

In the following we construct a Gel'fand triple $[\mathcal{E}]_u \subset (L^2) \subset [\mathcal{E}]_u^*$ associated with $u \in C_{+,1/2}$ satisfying conditions (U0) (U2) (U3) and discuss characterization theorems for test and generalized functions under the same condition.

Note that condition (U0) is merely a normalization condition and is equivalent to $\ell_u(0) = 1$, which guarantees the condition $\alpha(0) = 1$ in (A1). Obviously, (U1) is stronger than condition (U0). However we have the following lemma:

Lemma 3.1. *For $u \in C_{+,1/2}$, there exists a minimum point $\underline{r}$ of u and a maximum point $\bar{r}$ of u on $[0, \underline{r}]$, i.e. $u(\bar{r}) = \inf_{r\geq 0} u(r)$ and $u(\bar{r}) = \sup_{0\leq r\leq \underline{r}} u(r)$. Define*

$$v(r) = \begin{cases} u(\underline{r}) \text{ for } 0 \leq r \leq \underline{r} \\ u(r) \text{ for } \underline{r} \leq r. \end{cases}$$

Then v belongs to $C_{+,1/2}$ and equivalent to u;

$$v(r) \leq u(r) \leq \frac{u(\bar{r})}{u(\underline{r})} v(r).$$

Moreover

$$\ell_v(t) = \ell_u(t) \quad \text{for} \quad t \geq 0 \quad \text{and} \quad \mathcal{L}_v(r) = \mathcal{L}_u(r) \quad \text{for} \quad r \geq 0.$$

If u satisfies (U0) and (U3), then v satisfies (U1) and (U3) with $v(0) = u(\underline{r}) = 1$ and $u(\bar{r}) = u(0)$.

Proof. Since r^{-t} is decreasing on $[0,\ \underline{r}]$ for a fixed $t \geq 0$ and $u(r)$ takes the minimum value at $\underline{r}$,

$$\inf_{0<r\leq \underline{r}} u(r)r^{-t} = u(\underline{r})\underline{r}^{-t} = v(\underline{r})\underline{r}^{-t} = \inf_{0<r\leq \underline{r}} v(r)r^{-t}.$$

This implies $\ell_v(t) = \ell_u(t)$ for $t \geq 0$. Other statements are obvious from above. $\square$

If u satisfies (U0) and (U3), then by Fact 2.1, this v is $(\log, \exp)$-convex and hence by Fact 2.5 v is equivalent to $\mathcal{L}_v$. This means that u is equivalent to $\mathcal{L}_u$ by the above lemma.

Now we will construct a Gel'fand triple $[\mathcal{E}]_u \subset (L^2) \subset [\mathcal{E}]_u^*$ associated with a fixed function $u \in C_{+,1/2}$ satisfying conditions (U0) (U2) (U3). First we will relate the Gel'fand triple $[\mathcal{E}]_u \subset (L^2) \subset [\mathcal{E}]_u^*$ to a CKS-space. By (U3) and Fact 2.1, we have a log-convex function $\{\ell_u(t)\}$. Define log-concave sequence α_u like (2.4) by

$$\alpha_u(n) = \frac{1}{n!\ell_u(n)}. \tag{3.1}$$

Thus, by Lemmas 2.2 and 2.3 we can construct a Gel'fand triple $[\mathcal{E}]_{u_\alpha} \subset (L^2) \subset [\mathcal{E}]_{u_\alpha}^*$, which is denoted by $[\mathcal{E}]_u \subset (L^2) \subset [\mathcal{E}]_u^*$. Norms for $p > 0$ are also denoted as

$$\|\varphi\|_{p,u} = \|\varphi\|_{p,\alpha_u} = \left(\sum_{n=0}^{\infty} \frac{1}{\ell_u(n)} |f_n|_p^2 \right)^{1/2} \tag{3.2}$$

for $\varphi = \sum_{n=0}^{\infty} \langle :\cdot^{\otimes n}:, f_n \rangle \in [\mathcal{E}]_u$ and

$$\|\Phi\|_{-p,(u)} = \|\Phi\|_{-p,1/\alpha_u}. = \left(\sum_{n=0}^{\infty} \ell_u(n)(n!)^2 |f_n|_{-p}^2 \right)^{1/2} \tag{3.3}$$

for $\Phi = \sum_{n=0}^{\infty} \langle :\cdot^{\otimes n}:, f_n \rangle \in [\mathcal{E}]_u^*$. Subspaces are denoted as

$$[\mathcal{E}_p]_u = [\mathcal{E}_p]_\alpha \quad \text{and} \quad [\mathcal{E}_p]_u^* = [\mathcal{E}_p]_\alpha^* \quad \text{for} \quad p > 0.$$

Applying Theorem 1.1, we can show the following theorem:

Theorem 3.2. *Suppose $u \in C_{+,1/2}$ satisfies conditions (U0) (U2) (U3). Then*
(i) There exist positive constants c and a such that for $\Phi \in [\mathcal{E}]_u^*$

$$|(S\Phi)(\xi)| \leq \|\Phi\|_{-p,(u)} \sqrt{c}\, u^*\big(a|\xi|_p^2\big)^{1/2}, \qquad \xi \in \mathcal{E}_c. \tag{3.4}$$

(ii) A complex-valued function F on $\mathcal{E}_c$ is the S-transform of a generalized function $\Phi \in [\mathcal{E}]_u^*$ if and only if it satisfies the conditions:

1. *For any $\xi, \eta \in \mathcal{E}_c$, the function $F(z\xi + \eta)$ is an entire function of $z \in \mathbb{C}$.*
2. *There exist constants $K, a, p \geq 0$ such that*

$$|F(\xi)| \leq K u^*\big(a|\xi|_p^2\big)^{1/2}, \qquad \xi \in \mathcal{E}_c.$$

(iii) In the above case, for any $q > p$ such that $ae^2\|i_{q,p}\|_{HS}^2 < 1$, we have the inequality:

$$\|\Phi\|_{-q,(u)} \leq K \left(1 - ae^2\|i_{q,p}\|_{HS}^2\right)^{-1/2}. \tag{3.5}$$

Remark. The growth condition (2) is equivalent to the condition: There exist constants $K, p \geq 0$ such that

$$|F(\xi)| \leq K u^*\big(|\xi|_p^2\big)^{1/2}, \qquad \xi \in \mathcal{E}_c.$$

Proof. By Equation (1.5) and Lemma 2.3, we have

$$\|:e^{\langle \cdot, \xi \rangle}:\|_{p,u} = \mathcal{L}_u^{\#}\big(|\xi|_p^2\big)^{1/2} \quad p \geq 0.$$

Since $G_{\alpha_u}(r) = \mathcal{L}_u^{\#}(r)$. is equivalent to u^* by Lemma 2.3,

$$|S\Phi(\xi)| = |\langle \Phi, :e^{\langle \cdot, \xi \rangle}: \rangle| \leq \|\Phi\|_{-p,(u)} \mathcal{L}_u^{\#}\big(|\xi|_p^2\big)^{1/2} \leq \|\Phi\|_{-p,(u)} \sqrt{c} u^*\big(a|\xi|_p^2\big)^{1/2}$$

with suitable $c, a > 0$. Thus we see (i). By Lemma 2.4, $\{\alpha(n)\}$ satisfies condition near-$(B2)$. By Fact 2.9, (ii) follows from Theorem 1.1.

We can prove the estimation of the norm (3.5) with the same idea as in the proof of Theorem 8.2 in [21]. Φ and $F = S\Phi$ are expanded as $\Phi = \sum_{n=0}^{\infty} \langle : \cdot^{\otimes n} :, f_n \rangle$. $F(\xi) = \displaystyle\sum_{n=0}^{\infty} \langle f_n, \xi^{\otimes n} \rangle$, respectively. For any $\xi_1, \ldots, \xi_n \in \mathcal{E}_c$, applying the Cauchy formula to $F(z_1\xi_1 + \cdots + z_n\xi_n)$, we have

$$|\langle f_n, \xi_1 \widehat{\otimes} \cdots \widehat{\otimes} \xi_n \rangle| \leq \frac{K}{n!} a^{n/2} n^n \left(\frac{u^*(r)}{r^n} \right)^{1/2} |\xi_1|_p \cdots |\xi_n|_p.$$

Take the infimum over $r > 0$ and use the definition of the Legendre transform. Then, we conclude that

$$|f_n|_{-q}^2 \leq \frac{K^2}{(n!)^2} a^n n^{2n} \ell_{u^*}(n) \|i_{q,p}\|_{HS}^{2n}. \tag{3.6}$$

Then by Equations (3.3) and (3.6),

$$\|\Phi\|_{-q,(u)}^2 = \sum_{n=0}^{\infty} \ell_u(n)(n!)^2 |f_n|_{-q}^2 \leq K^2 \sum_{n=0}^{\infty} \ell_u(n) a^n n^{2n} \ell_{u^*}(n) \|i_{q,p}\|_{HS}^{2n}.$$

However by Fact 2.7, we have $\ell_{u^*}(n) = \ell_u(n)^{-1} n^{-2n} e^{2n}$. Therefore,

$$\|\Phi\|_{-q,(u)}^2 \leq K^2 \sum_{n=0}^{\infty} a^n e^{2n} \|i_{q,p}\|_{HS}^{2n} \leq K^2 \left(1 - ae^2 \|i_{q,p}\|_{HS}^2\right)^{-1}$$

by the assumption $ae^2 \|i_{q,p}\|_{HS}^2 < 1$. (Of course, this estimation implies (ii), directly.) $\square$

Proposition 3.3. *(i) For $\Phi = \sum_{n=0}^{\infty} \langle : \cdot^{\otimes n} :, f_n \rangle$ and $p \geq 0$, we define a new norm*

$$\|\Phi\|_{-p,u^*} = \left(\sum_{n=0}^{\infty} \frac{1}{\ell_{u^*(n)}} |f_n|_{-p}^2 \right)^{1/2}. \tag{3.7}$$

For any $p \geq 0$ and $q \geq p + \frac{\log 2}{2\log(\rho^{-1})}$, we have

$$e^{-1} \|\Phi\|_{-q,(u)} \leq \|\Phi\|_{-p,u^*} \leq \|\Phi\|_{-p,(u)}, \qquad \forall \Phi \in [\mathcal{E}_p]_u^*. \tag{3.8}$$

In particular, if $S\Phi(\xi)$ satisfies the conditions (1) and (2) in Theorem 3.2, we have

$$\|\Phi\|_{-q,u^*} \leq K \left(1 - ae^2 \|i_{q,p}\|_{HS}^2\right)^{-1/2}.$$

(ii) For a test function $\varphi = \sum_{n=0}^{\infty} \langle : \cdot^{\otimes n} :, f_n \rangle$ and $p \geq 0$, a new norm corresponding to the norm $\|\cdot\|_{-p,u^*}$ in Equation (3.7) is given by

$$\|\varphi\|_{p,(u^*)} = \left(\sum_{n=0}^{\infty} \ell_{u^*(n)} (n!)^2 |f_n|_p^2 \right)^{1/2}. \tag{3.9}$$

We have the corresponding inequalities, i.e., for any $p \geq 0$ and $q \geq p + \frac{\log 2}{2\log(\rho^{-1})}$,

$$\|\varphi\|_{p,u} \leq \|\varphi\|_{p,(u^*)} \leq e \|\varphi\|_{q,u}, \qquad \forall \varphi \in [\mathcal{E}_q]_u.$$

Proof. First we point out the following inequalities from page 357 in [21]

$$e^{-1}2^{-n/2}n! \le \left(\frac{n}{e}\right)^n \le n!. \tag{3.10}$$

By Fact 2.7 and the second inequality in Equation (3.10),

$$\|\Phi\|^2_{-p,u^*} = \sum_{n=0}^{\infty} \ell_u(n) \left(\frac{n}{e}\right)^{2n} |f_n|^2_{-p} \le \sum_{n=0}^{\infty} \ell_u(n)(n!)^2 |f_n|^2_{-p} = \|\Phi\|^2_{-p,(u)}.$$

This gives the second inequality in Equation (3.8). On the other hand, we can use the first inequality in Equation (3.10) to get

$$\|\Phi\|^2_{-p,u^*} \ge e^{-2} \sum_{n=0}^{\infty} \ell_u(n) 2^{-n} (n!)^2 |f_n|^2_{-p}.$$

Note that $|f|_{-p} \ge \rho^{p-q}|f|_{-q}$ for any $q \ge p$ and $f \in \mathcal{E}'_p$. Therefore,

$$\|\Phi\|^2_{-p,u^*} \ge e^{-2} \sum_{n=0}^{\infty} \ell_u(n)(n!)^2 \left(2^{-1}\rho^{-2(q-p)}\right)^n |f_n|^2_{-q}.$$

When $2\rho^{2(q-p)} \le 1$, i.e., $q \ge p + \frac{\log 2}{2\log(\rho^{-1})}$, the above inequality yields that

$$\|\Phi\|^2_{-p,u^*} \ge e^{-2} \sum_{n=0}^{\infty} \ell_u(n)(n!)^2 |f_n|^2_{-q} = e^{-2}\|\Phi\|^2_{-q,(u)}.$$

This proves the first inequality in Equation (3.8). The assertions to test functions are proved similarly. □

Next we consider the characterization of test functions.

Theorem 3.4. *Suppose $u \in C_{+,1/2}$ satisfies conditions (U0) (U2) (U3). Then*
(i) There exists a positive constant a such that for $\varphi \in [\mathcal{E}]_u$,

$$|S\varphi(\xi)| \le \|\varphi\|_{u,p} \mathcal{L}_u\big(|\xi|_p^2\big) \le \|\varphi\|_{u,p} \sqrt{\frac{2e}{\log 2}}\, u\big(a|\xi|_p^2\big).$$

(ii) A complex-valued function F on $\mathcal{E}_c$ is the S-transform of a test function $\varphi \in [\mathcal{E}]_u$ if and only if it satisfies the conditions:

1. *For any $\xi, \eta \in \mathcal{E}_c$, the function $F(z\xi + \eta)$ is an entire function of $z \in \mathbb{C}$.*
2. *There exist constants $K, a, p \ge 0$ such that*

$$|F(\xi)| \le K u\big(a|\xi|^2_{-p}\big)^{1/2}, \qquad \xi \in \mathcal{E}_c. \tag{3.11}$$

(iii) Let $q \in [0, p)$ be a number such that $ae^2\|i_{p,q}\|^2_{HS} < 1$. Then F is the S-transform of $\varphi \in [\mathcal{E}_q]_u$ and

$$\|\varphi\|_{q,u} \le K \left(1 - ae^2\|i_{p,q}\|^2_{HS}\right)^{-1/2}. \tag{3.12}$$

Remark. The growth condition (2) is equivalent to the condition: For any $p \ge 0$ there exists a constant $K \ge 0$ such that

$$|F(\xi)| \le K u\big(|\xi|^2_{-p}\big)^{1/2}, \qquad \xi \in \mathcal{E}_c.$$

Proof. Let v be as in Lemma 3.1. It is easy to see that $G_{1/\alpha_u}(r) = \mathcal{L}_u(r) = \mathcal{L}_v(r)$ and hence
$$\|:e^{\langle\cdot,\xi\rangle}:\|_{p,(u)} = \mathcal{L}_u\big(|\xi|_{-p}^2\big)^{1/2} \quad p \geq 0$$
by Equation (1.7). Therefore
$$|S\varphi(\xi)| = |\langle\Phi, :e^{\langle\cdot,\xi\rangle}:\rangle| \leq \|\varphi\|_{p,u}\mathcal{L}_u\big(|\xi|_{-p}^2\big)^{1/2}.$$
By using Fact 2.5 (1) for v with $a = 2$, we have
$$\mathcal{L}_u(r) = \mathcal{L}_v(r) \leq \frac{2e}{\log 2}v(2r) \leq \frac{2e}{\log 2}u(2r).$$
Thus we see (i).

By using Fact 2.5 (2) for v with $k = 2$, we have
$$u(r) \leq u(0)v(r) \leq Cu(0)\mathcal{L}_v(2^2 r) = Cu(0)\mathcal{L}_u(2^2 r).$$
We have already seen $\mathcal{L}_u(r) \leq \frac{2e}{\log 2}u(2r)$ in the proof of (i). Thus u and $\mathcal{L}_u = G_{1/\alpha_u}$ are equivalent. Due to Lemma 2.5, $\{\alpha_u(n)\}$ satisfies the condition $(\widetilde{\text{B}}2)$. By Theorem 1.2, we can prove (ii).

The proof of (iii) is similar to that of Theorem 3.2 (iii). The key of the proof is the estimation
$$|f_n|_p \leq \inf_{r>0} \frac{u(an^2r^2)^{1/2}}{r^n} = a^{n/2}n^n\ell_u(n)^{1/2} \leq (a^{1/2}e)^n n!\ell_u(n)^{1/2}$$
for $F(\xi) = \sum_{n=0}^{\infty}\langle f_n, \xi^{\otimes n}\rangle$. This implies (iii). □

The following examples can be applicable to Theorems 3.2 and 3.4.

Example 3.1. Consider
$$u(r) = u^*(r) = e^r.$$
Then it is obvious to check that consditions (U0) (U2) (U3) are satisfied.

Example 3.2. For $0 \leq \beta < 1$, let u be the function defined by
$$u(r) = \exp\left[(1+\beta)r^{\frac{1}{1+\beta}}\right].$$
It is easy to check that u belongs to $C_{+,1/2}$ and satisfies conditions (U0) (U2) (U3). By Example 4.3 in [4], the dual Legendre transform u^* of u is given by
$$u^*(r) = \exp\left[(1-\beta)r^{\frac{1}{1-\beta}}\right].$$
Hence Theorems 3.2 and 3.4 can be applied to the Gel'fand triple $[\mathcal{E}]_u \subset (L^2) \subset [\mathcal{E}]_u^*$ for the pair of functions u^* and u.

Example 3.3. Consider the function $v(r) = \exp\big[e^r - 1\big]$. Obviously, $v \in C_{+,1/2}$. Let $u = v^*$ be the dual Legendre transform of v. Then $u(0) = \sup_{s\geq 0} v(s)^{-1} = 1$ and by Fact 2.6 u belongs to $C_{+,1/2}$ and is an increasing (log, x^2)-convex function on $[0,\infty)$. Hence $u \in C_{+,1/2}$ satisfies conditions (U1) and (U3). It is shown in Example 4.4 in [4] that u is equivalent to the function
$$w(r) = \exp\left[2\sqrt{r\log\sqrt{r}}\right]. \tag{3.13}$$

Obviously, w satisfies condition (U2) and so u also satisfies condition (U2). On the other hand, we have $u^* = (v^*)^* = v$ by Fact 2.8. Hence Theorems 3.2 and 3.4 can be applied to the Gel'fand triple $[\mathcal{E}]_u \subset (L^2) \subset [\mathcal{E}]_u^*$ for the following pair of functions:

$$u^*(r) = \exp\left[e^r - 1\right], \quad u(r) = (u^*)^*.$$

Observe that there is no exact form for the function u. Thus for Theorems 3.4 we should use the equivalent function w in Equation (3.13) as the growth function.

In general, let $\exp_k(r) = \exp(\exp(\cdots(\exp(r))))$ be the k-th iterated exponential function and consider the function

$$v_k(r) = \frac{\exp_k(r)}{\exp_k(0)}.$$

The dual Legendre transform $u_k = v_k^*$ belongs to $C_{+,1/2}$ and satisfies conditions (U1) (U2) (U3). The function u_k is equivalent to the function w_k given in Equation (1.11), i.e.,

$$w_k(r) = \exp\left[2\sqrt{r \log_{k-1}\sqrt{r}}\,\right]. \tag{3.14}$$

We have $u_k^* = v_k$ and Theorems 3.2, and 3.4 can be applied to the Gel'fand triple $[\mathcal{E}]_{u_k} \subset (L^2) \subset [\mathcal{E}]_{u_k}^*$ for the following pair of functions:

$$u_k^*(r) = \frac{\exp_k(r)}{\exp_k(0)}, \quad u_k(r) = (u_k^*)^*.$$

Again there is no exact form for the function u_k. Thus for Theorems 3.4 we should use the equivalent function w_k in Equation (3.14) as the growth function.

4. Intrinsic topology and Hida measures

In the space $[\mathcal{E}]_u$ of test functions there are two families of norms, namely, $\{\|\cdot\|_{p,u};\, p \geq 0\}$ defined in Equation (3.2) and $\{\|\cdot\|_{p,(u^*)};\, p \geq 0\}$ defined in Equation (3.9). As we pointed out in the Remark of Lemma 3.3, these two families are equivalent. Observe that both $\|\varphi\|_{p,u}$ and $\|\varphi\|_{p,(u^*)}$ are defined in terms of the Wiener-Itô expansion of φ.

In this section we will introduce another equivalent family of norms on $[\mathcal{E}]_u$, i.e., $\{\|\cdot\|_{\mathcal{A}_{p,u}};\, p \geq 0\}$. This family of norms is intrinsic in the sense that $\|\varphi\|_{\mathcal{A}_{p,u}}$ is defined directly in terms of the analyticity and growth condition of φ.

First the analyticity, each test function φ in $[\mathcal{E}]_u$ has a unique analytic extension (see §6.3 of the book [21]) given by

$$\varphi(x) = \langle\!\langle :e^{\langle\cdot,x\rangle}:, \Theta\varphi \rangle\!\rangle, \qquad x \in \mathcal{E}_c', \tag{4.1}$$

where Θ is the unique linear operator taking $e^{\langle\cdot,\xi\rangle}$ into $:e^{\langle\cdot,\xi\rangle}:$ for all $\xi \in \mathcal{E}_c$. This operator turns out to be the same as $\mathcal{G}_{i,1}$ defined in Equation (5.3) in section 5. By Theorem 5.6 this operator is continuous from $[\mathcal{E}]_u$ into itself.

Now, let $p \geq 0$ be any fixed number. Choose $p_1 > p$ such that $2\rho^{2(p_1-p)} \leq 1$. Then use Equations (4.1) and (3.11) to get

$$|\varphi(x)| \leq \|\Theta\varphi\|_{p_1,u}\, \|:e^{\langle\cdot,x\rangle}:\|_{-p_1,(u)} \leq \|\Theta\varphi\|_{p_1,u} \sqrt{\frac{2e}{\log 2}}\, u\big(2|x|_{-p_1}^2\big)^{1/2}.$$

Note that $2|x|_{-p_1}^2 \le 2\rho^{2(p_1-p)}|x|_{-p}^2 \le |x|_{-p}^2$ by the above choice of p_1. Since u is an increasing function, we see that

$$|\varphi(x)| \le \|\Theta\varphi\|_{p_1,u}\sqrt{\frac{2e}{\log 2}}\,u\big(|x|_{-p}^2\big)^{1/2}.$$

However Θ is a continuous linear operator from $[\mathcal{E}]_u$ into itself. Hence there exist positive constants q and $K_{p,q}$ such that $\|\Theta\varphi\|_{p_1,u} \le K_{p,q}\|\varphi\|_{q,u}$. Therefore,

$$|\varphi(x)| \le C_{p,q}\|\varphi\|_{q,u}\,u\big(|x|_{-p}^2\big)^{1/2}, \qquad x \in \mathcal{E}'_{p,c}, \tag{4.2}$$

where $C_{p,q} = K_{p,q}\sqrt{2e/\log 2}$. This is the growth condition for test functions.

Being motivated by Equation (4.2), we define

$$\|\varphi\|_{\mathcal{A}_{p,u}} = \sup_{x\in\mathcal{E}'_{p,c}} |\varphi(x)|\,u\big(|x|_{-p}^2\big)^{-1/2}. \tag{4.3}$$

Obviously, $\|\cdot\|_{\mathcal{A}_{p,u}}$ is a norm on $[\mathcal{E}]_u$ for each $p \ge 0$. The next theorem generalizes the results of Kuo [21] and Lee [25].

Theorem 4.1. *Suppose $u \in C_{+,1/2}$ satisfies conditions (U0) (U2) (U3). Then the families of norms $\{\|\cdot\|_{\mathcal{A}_{p,u}};\, p \ge 0\}$ and $\{\|\cdot\|_{p,u};\, p \ge 0\}$ are equivalent, i.e., they generate the same topology on $[\mathcal{E}]_u$.*

Remark. (1) This theorem has been announced in [6], but (U1) condition was assumed there instead of (U0). This theorem can be interpreted in another way which gives an alternative construction of test functions.

(2) For $p \ge 0$, let $\mathcal{A}_{p,u}$ consist of all functions φ on $\mathcal{E}'_c$ satisfying the conditions:

(a) φ is an analytic function on $\mathcal{E}'_{p,c}$.

(b) There exists a constant $C \ge 0$ such that

$$|\varphi(x)| \le Cu\big(|x|_{-p}^2\big)^{1/2}, \qquad \forall x \in \mathcal{E}'_{p,c}.$$

(3) Suppose that v is equivalent to u. Then, it is obvious to see that the family of norms $\{\|\cdot\|_{\mathcal{A}_{p,v}};\ p \ge 0\}$ and $\{\|\cdot\|_{\mathcal{A}_{p,u}};\ p \ge 0\}$ are equivalent.

For each $\varphi \in \mathcal{A}_{p,u}$, define $\|\varphi\|_{\mathcal{A}_{p,u}}$ by Equation (4.3). Then $\mathcal{A}_{p,u}$ is a Banach space with norm $\|\cdot\|_{\mathcal{A}_{p,u}}$. Let $\mathcal{A}_u$ be the projective limit of $\{\mathcal{A}_{p,u};\, p \ge 0\}$. We can use the above theorem to conclude that $\mathcal{A}_u = [\mathcal{E}]_u$ as vector spaces with the same topology. Here the equality $\mathcal{A}_u = [\mathcal{E}]_u$ requires the analytic extension of a test function $\varphi \in [\mathcal{E}]_u$ in Equation (4.1).

Proof. Let $p \ge 0$ be any given number. We have already shown that there exist constants $q > p$ and $C_{p,q} \ge 0$ such that Equation (4.2) holds. It follows that

$$\|\varphi\|_{\mathcal{A}_{p,u}} = \sup_{x\in\mathcal{E}'_{p,c}} |\varphi(x)|\,u\big(|x|_{-p}^2\big)^{-1/2} \le C_{p,q}\|\varphi\|_{q,u}.$$

Hence for any $p \ge 0$, there exist constants $q > p$ and $C_{p,q} \ge 0$ such that

$$\|\varphi\|_{\mathcal{A}_{p,u}} \le C_{p,q}\|\varphi\|_{q,u}, \qquad \forall \varphi \in [\mathcal{E}]_u. \tag{4.4}$$

To show the converse, first note that by condition (U2) there exist constants $c_1, c_2 > 0$ such that $u(r) \le c_1 e^{c_2 r}$, $r \ge 0$. Next note that by Fernique's theorem (see [9] [20] or page 328 in [21]) we have

$$\int_{\mathcal{E}'} e^{2c_2|x|_{-\lambda}^2}\,d\mu(x) < \infty \qquad \text{for all large } \lambda.$$

Now, let $p \geq 0$ be any given number. Choose $q > p$ large enough such that

$$4e^2\|i_{q,p}\|_{HS}^2 < 1, \qquad \int_{\mathcal{E}'} e^{2c_2|x|_{-q}^2}\, d\mu(x) < \infty. \tag{4.5}$$

With this choice of q we will show below that

$$\|\varphi\|_{p,u} \leq L_{p,q}\|\varphi\|_{\mathcal{A}_{q,u}}, \qquad \forall \varphi \in [\mathcal{E}]_u, \tag{4.6}$$

where $L_{p,q}$ is the constant given by

$$L_{p,q} = \sqrt{c_1}\left(1 - 4e^2\|i_{q,p}\|_{HS}^2\right)^{-1/2} \int_{\mathcal{E}'} e^{2c_2|x|_{-q}^2}\, d\mu(x). \tag{4.7}$$

Observe that the theorem follows from Equations (4.4) and (4.6).

To prove Equation (4.6), let $\varphi \in [\mathcal{E}]_u$ and $F = S\varphi$. Then F can be written as an integral (see page 36 in [21])

$$F(\xi) = \int_{\mathcal{E}'} \varphi(x+\xi)\, d\mu(x), \qquad \xi \in \mathcal{E}_c.$$

Hence for the above choice of q, we have

$$\begin{aligned} |F(\xi)| &\leq \int_{\mathcal{E}'} |\varphi(x+\xi)|\, d\mu(x) \\ &\leq \int_{\mathcal{E}'} \Big(|\varphi(x+\xi)|\, u\big(|x+\xi|_{-q}^2\big)^{-1/2}\Big)\, u\big(|x+\xi|_{-q}^2\big)^{1/2}\, d\mu(x) \\ &\leq \|\varphi\|_{\mathcal{A}_{q,u}} \int_{\mathcal{E}'} u\big(|x+\xi|_{-q}^2\big)^{1/2}\, d\mu(x). \end{aligned}$$

However by condition (U0), $u \geq 1$ on $[0, \infty)$. Hence $u(r)^{1/2} \leq u(r)$ for all $r \geq 0$. Therefore,

$$|F(\xi)| \leq \|\varphi\|_{\mathcal{A}_{q,u}} \int_{\mathcal{E}'} u\big(|x+\xi|_{-q}^2\big)\, d\mu(x). \tag{4.8}$$

By condition (U3), u is (log, x^2)-convex. Thus in particular, we have

$$u\left(\big(\tfrac{1}{2}r_1 + \tfrac{1}{2}r_2\big)^2\right) \leq u\big(r_1^2\big)^{1/2}\, u\big(r_2^2\big)^{1/2}, \qquad \forall r_1, r_2 \geq 0.$$

Put $r_1 = 2|x|_{-q}$ and $r_2 = 2|\xi|_{-q}$ to get

$$\begin{aligned} u\big(|x+\xi|_{-q}^2\big) &\leq u\left(\big(\tfrac{1}{2}2|x|_{-q} + \tfrac{1}{2}2|\xi|_{-q}\big)^2\right) \\ &\leq u\big(4|x|_{-q}^2\big)^{1/2}\, u\big(4|\xi|_{-q}^2\big)^{1/2}. \end{aligned}$$

Then integrate over $\mathcal{E}'$ to obtain the inequality:

$$\int_{\mathcal{E}'} u\big(|x+\xi|_{-q}^2\big)\, d\mu(x) \leq u\big(4|\xi|_{-q}^2\big)^{1/2} \int_{\mathcal{E}'} u\big(4|x|_{-q}^2\big)^{1/2}\, d\mu(x). \tag{4.9}$$

Put Equation (4.9) into Equation (4.8) to get

$$|F(\xi)| \leq \|\varphi\|_{\mathcal{A}_{q,u}} u\big(4|\xi|_{-q}^2\big)^{1/2} \int_{\mathcal{E}'} u\big(4|x|_{-q}^2\big)^{1/2}\, d\mu(x). \tag{4.10}$$

Now, by the inequality $u(r) \leq c_1 e^{c_2 r}$, we have

$$\int_{\mathcal{E}'} u\big(4|x|_{-q}^2\big)^{1/2}\, d\mu(x) \leq \sqrt{c_1} \int_{\mathcal{E}'} e^{2c_2|x|_{-q}^2}\, d\mu(x), \tag{4.11}$$

which is finite by the choice of q in Equation (4.5).

By Equations (4.10) and (4.11), we see that

$$|F(\xi)| \leq \|\varphi\|_{\mathcal{A}_{q,u}} \sqrt{c_1} \left(\int_{\mathcal{E}'} e^{2c_2|x|_{-q}^2} \, d\mu(x) \right) u\big(4|\xi|_{-q}^2\big)^{1/2}, \qquad \xi \in \mathcal{E}_c.$$

With this inequality and the choice of q in Equation (4.5) we can apply Theorem 3.4 to show that for any $\varphi \in [\mathcal{E}]_u$,

$$\|\varphi\|_{p,u} \leq L_{p,q} \|\varphi\|_{\mathcal{A}_{q,u}},$$

where $L_{p,q}$ is given by Equation(4.7). Thus the inequality in Equation (4.6) holds and so the proof is completed. □

Next, we consider the characterization of Hida measures. However first we need to prepare two lemmas.

Lemma 4.2. *Suppose $u \in C_{+,\log}$ is (log, x^k)-convex. Then*

$$\mathcal{L}_u(r)^2 \leq \ell_u(0) \mathcal{L}_u\big(2^{k+1} r\big), \qquad \forall r \in [0, \infty). \tag{4.12}$$

Remark. Note that $\mathcal{L}_u(r) \geq \ell_u(0)$ for all $r \geq 0$. Hence we have inequalities

$$\ell_u(0)\mathcal{L}_u(r) \leq \mathcal{L}_u(r)^2 \leq \ell_u(0)\mathcal{L}_u\big(2^{k+1} r\big), \qquad \forall r \in [0, \infty).$$

Thus $\mathcal{L}_u$ and $\mathcal{L}_u^2$ are equivalent for any (log, x^k)-convex function $u \in C_{+,\log}$. It follows that u and u^2 are equivalent for such a function u.

Proof. Apply Fact 2.4 (2) to get

$$\begin{aligned}
\mathcal{L}_u(r)^2 &= \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \ell_u(j)\ell_u(m) r^{j+m} \\
&\leq \ell_u(0) \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} 2^{k(j+m)} \ell_u(j+m) r^{j+m} \\
&= \ell_u(0) \sum_{j=0}^{\infty} \sum_{n=j}^{\infty} 2^{kn} \ell_u(n) r^n.
\end{aligned}$$

Then change the order of summation and use the inequality $n+1 \leq 2^n$ to get

$$\begin{aligned}
\mathcal{L}_u(r)^2 &\leq \ell_u(0) \sum_{n=0}^{\infty} (n+1) 2^{kn} \ell_u(n) r^n \\
&\leq \ell_u(0) \sum_{n=0}^{\infty} 2^{(k+1)n} \ell_u(n) r^n \\
&= \ell_u(0) \mathcal{L}_u\big(2^{k+1} r\big).
\end{aligned}$$

□

Lemma 4.3. *Suppose $u \in C_{+,\log}$ is increasing and (log, x^k)-convex. Then for any $a > 1$, we have*

$$\mathcal{L}_u(r) \leq \sqrt{\ell_u(0) \frac{ea}{\log a}} \, u\big(a 2^{k+1} r\big)^{1/2}. \tag{4.13}$$

Proof. Recall a fact mentioned in the beginning of section 2 that if u is increasing and (log, x^k)-convex, then u is (log, exp)-convex. Hence this lemma follows from Lemma 4.2 and Fact 2.5 (1). □

A measure ν on $\mathcal{E}'$ is called a *Hida measure* associated with u if $[\mathcal{E}]_u \subset L^1(\nu)$ and the linear functional $\varphi \mapsto \int_{\mathcal{E}'} \varphi(x)\, d\nu(x)$ is continuous on $[\mathcal{E}]_u$. In this case, ν induces a generalized function, denoted by $\widetilde{\nu}$, in $[\mathcal{E}]_u^*$ such that

$$\langle\!\langle \widetilde{\nu}, \varphi \rangle\!\rangle = \int_{\mathcal{E}'} \varphi(x)\, d\nu(x), \qquad \varphi \in [\mathcal{E}]_u. \tag{4.14}$$

The next theorem generalizes the results of Kuo [21] and Lee [25].

Theorem 4.4. *Suppose $u \in C_{+,1/2}$ satisfies conditions (U0) (U2) (U3). Then a measure ν on $\mathcal{E}'$ is a Hida measure with $\widetilde{\nu} \in [\mathcal{E}]_u^*$ if and only if ν is supported by $\mathcal{E}'_p$ for some $p \geq 0$ and*

$$\int_{\mathcal{E}'_p} u\big(|x|_{-p}^2\big)^{1/2}\, d\nu(x) < \infty. \tag{4.15}$$

Remark. This theorem has also been announced in [6], but the conditions (U1) (U2) (U3) were assumed.

Proof. To prove the sufficiency, suppose ν is supported by $\mathcal{E}'_p$ for some $p \geq 0$ and Equation (4.15) holds. Then for any $\varphi \in [\mathcal{E}]_u$,

$$\begin{aligned}
\int_{\mathcal{E}'} |\varphi(x)|\, d\nu(x) &= \int_{\mathcal{E}'_p} |\varphi(x)|\, d\nu(x) \\
&= \int_{\mathcal{E}'_p} \Big(|\varphi(x)| u\big(|x|_{-p}^2\big)^{-1/2} \Big)\, u\big(|x|_{-p}^2\big)^{1/2}\, d\nu(x) \\
&\leq \|\varphi\|_{\mathcal{A}_{p,u}} \int_{\mathcal{E}'_p} u\big(|x|_{-p}^2\big)^{1/2}\, d\nu(x).
\end{aligned} \tag{4.16}$$

By Theorem 4.1, $\{\|\cdot\|_{\mathcal{A}_{p,u}};\, p \geq 0\}$ and $\{\|\cdot\|_{p,u};\, p \geq 0\}$ are equivalent. Hence Equation (4.16) implies that $[\mathcal{E}]_u \subset L^1(\nu)$ and the linear functional

$$\varphi \longmapsto \int_{\mathcal{E}'} \varphi(x)\, d\nu(x), \qquad \varphi \in [\mathcal{E}]_u,$$

is continuous on $[\mathcal{E}]_u$. Thus ν is a Hida measure with $\widetilde{\nu}$ in $[\mathcal{E}]_u^*$.

To prove the necessity, suppose ν is a Hida measure inducing a generalized function $\widetilde{\nu} \in [\mathcal{E}]_u^*$. Then for all $\varphi \in [\mathcal{E}]_u$,

$$\langle\!\langle \widetilde{\nu}, \varphi \rangle\!\rangle = \int_{\mathcal{E}'} \varphi(x)\, d\nu(x). \tag{4.17}$$

Since $\{\|\cdot\|_{\mathcal{A}_{p,u}};\, p \geq 0\}$ and $\{\|\cdot\|_{p,u};\, p \geq 0\}$ are equivalent, the linear functional $\varphi \mapsto \langle\!\langle \widetilde{\nu}, \varphi \rangle\!\rangle$ is continuous with respect to $\{\|\cdot\|_{\mathcal{A}_{p,u}};\, p \geq 0\}$. Hence there exist constants $K, q \geq 0$ such that for all $\varphi \in [\mathcal{E}]_u$,

$$\big|\langle\!\langle \widetilde{\nu}, \varphi \rangle\!\rangle\big| \leq K \|\varphi\|_{\mathcal{A}_{q,u}}. \tag{4.18}$$

Note that by continuity, Equations (4.17) and (4.18) also hold for all $\varphi \in \mathcal{A}_{q,u}$, which is defined in the Remark of Theorem 4.1.

Now, with this q, we define a function γ on $\mathcal{E}'_{q,c}$ by

$$\gamma(x) = \mathcal{L}_u\big(2^{-4}\langle x, x\rangle_{-q}\big), \qquad x \in \mathcal{E}'_{q,c},$$

where $\langle\cdot,\cdot\rangle_{-q}$ is the bilinear pairing on $\mathcal{E}'_{q,c}$. Obviously, γ is analytic on $\mathcal{E}'_{q,c}$. On the other hand, apply Lemma 4.3 with $a = k = 2$ to get

$$|\gamma(x)| \leq \mathcal{L}_u\big(2^{-4}|x|_{-q}^2\big) \leq \sqrt{\frac{2e}{\log 2}}\, u\big(|x|_{-q}^2\big)^{1/2}, \qquad \forall x \in \mathcal{E}'_{q,c}.$$

This shows that $\gamma \in \mathcal{A}_{q,u}$ and we have

$$\|\gamma\|_{\mathcal{A}_{q,u}} \leq \sqrt{\frac{2e}{\log 2}}.$$

Then apply Equation (4.18) to the function γ,

$$\big|\langle\langle \widetilde{\nu}, \gamma\rangle\rangle\big| \leq K\|\gamma\|_{\mathcal{A}_{q,u}} \leq K\sqrt{\frac{2e}{\log 2}}.$$

Therefore, from Equation (4.17) with $\varphi = \gamma$ we conclude that

$$\left|\int_{\mathcal{E}'} \gamma(x)\, d\nu(x)\right| \leq K\sqrt{\frac{2e}{\log 2}}. \tag{4.19}$$

Note that $\gamma(x) = \mathcal{L}_u\big(2^{-4}|x|_{-q}^2\big)$ for $x \in \mathcal{E}'$. Hence Equation (4.19) implies that

$$\int_{\mathcal{E}'} \mathcal{L}_u\big(2^{-4}|x|_{-q}^2\big)\, d\nu(x) < \infty.$$

However $u(r) \leq C\mathcal{L}_u(4r)$ from Fact 2.5 (2) with $k = 2$. Therefore,

$$\int_{\mathcal{E}'} u\big(2^{-6}|x|_{-q}^2\big)\, d\nu(x) < \infty.$$

Now, choose $p > q$ large enough such that $\rho^{2(p-q)} \leq 2^{-6}$. Then $|x|_{-p}^2 \leq 2^{-6}|x|_{-q}^2$. Recall that u is increasing. Hence

$$\int_{\mathcal{E}'} u\big(|x|_{-p}^2\big)\, d\nu(x) < \infty.$$

Note that $u(r) \geq 1$ and so $u(r)^{1/2} \leq u(r)$. Thus we conclude that

$$\int_{\mathcal{E}'} u\big(|x|_{-p}^2\big)^{1/2}\, d\nu(x) < \infty.$$

This inequality implies that ν is supported by $\mathcal{E}'_p$ and Equation (4.15) holds. □

Before closing this section, let us explain the relationship with [10]. The basic equalities are

$$u(r) = e^{2\theta(\sqrt{r})}, \quad u^*(r) = e^{2\theta^*(\sqrt{r})}$$

where $\theta^*(s) = \sup_{t>0}\{st - \theta(t)\}$ is adopted in [10]. In the following table we give the correspondence between our U-conditions and θ-conditions.

	u	θ
$(U0)$	$\inf_{r\geq 0} u(r) = 1$	$\inf_{r\geq 0} \theta(r) = 0$
$(U1)$	u is increasing and $u(0) = 1$	θ is increasing and $\theta(0) = 0$
$(U2)$	$\lim_{r\to\infty} \dfrac{\log u(r)}{r} < \infty$	$\lim_{r\to\infty} \dfrac{\theta(r)}{r^2} < \infty$
$(U3)$	u is (log, x^2)-convex	θ is convex

Our intrinsic topology is the same as their topology. However, we are interested in the equivalences between the intrinsic topologies and the Hilbertian topologies defined in section 3.

5. Comparison of Conditions with the CKS-space

In this section we will discuss the continuity of various operators and wick products. This matter is beyond the scope of results in [10].

Let us consider again a CKS-space $[\mathcal{E}]_\alpha \subset (L^2) \subset [\mathcal{E}]_\alpha^*$ associated with a sequence $\{\alpha(n)\}$ of positive real numbers. Due to the discussion in sections 3 and 4, we conclude that for a CKS-space it is reasonable to assume the *four essential conditions*: (A1), (A2), near-(B2), near-($\widetilde{\text{B}}$2).

On the other hand, in [16] the following three conditions are imposed in order to prove the continuity of various linear operators acting on the spaces $[\mathcal{E}]_\alpha$ and $[\mathcal{E}]_\alpha^*$:

(C1) There exists a constant c_1 such that for all $n \leq m$,

$$\alpha(n) \leq c_1^m \alpha(m).$$

(C2) There exists a constant c_2 such that for all n and m,

$$\alpha(n+m) \leq c_2^{n+m} \alpha(n)\alpha(m).$$

(C3) There exists a constant c_3 such that for all n and m,

$$\alpha(n)\alpha(m) \leq c_3^{n+m} \alpha(n+m).$$

It is shown in [16] that (C3) implies (C1). In the next two theorems we will show that conditions near-(B2) and near-($\widetilde{\text{B}}$2) imply conditions (C2) and (C3), respectively.

Theorem 5.1. *If a sequence $\{\alpha(n)\}$ of positive real numbers satisfies condition near-(B2) and $\alpha(0) \geq 1$, then it satisfies condition (C2).*

Proof. Since $\{\alpha(n)\}$ satisfies condition near-(B2), it is equivalent to a sequence $\{\lambda(n)\}$ of positive real numbers such that $\{\lambda(n)/n!\}$ is log-concave. Apply Equation (2.1) to the sequence $\beta(n) = \lambda(n)/\lambda(0)$. Then we get

$$\lambda(n+m) \leq \lambda(0)^{-1} 2^{n+m} \lambda(n)\lambda(m), \qquad \forall n, m \geq 0. \tag{5.1}$$

On the other hand, recall that $\{\alpha(n)\}$ and $\{\lambda(n)\}$ are equivalent. Hence there exist constants $K_1, K_2, c_1, c_2 > 0$ such that

$$K_1 c_1^n \lambda(n) \leq \alpha(n) \leq K_2 c_2^n \lambda(n). \tag{5.2}$$

From Equations (5.1) and (5.2) we can easily derive that

$$\alpha(n+m) \leq \lambda(0)^{-1} K_1^{-2} K_2 \big(2c_1^{-1}c_2\big)^{n+m} \alpha(n)\alpha(m), \qquad \forall n, m \geq 0.$$

Let $c = \max\big\{1,\, \lambda(0)^{-1}K_1^{-2}K_2,\, 2c_1^{-1}c_2\big\}$. Then the last inequality implies that

$$\alpha(n+m) \leq c^{2(n+m)} \alpha(n)\alpha(m), \qquad \forall n+m \geq 1.$$

However by assumption $\alpha(0) \geq 1$ and so this inequality also holds for $n = m = 0$. Thus the sequence $\{\alpha(n)\}$ satisfies condition (C2). □

Theorem 5.2. *If a sequence $\{\alpha(n)\}$ of positive real numbers satisfies condition near-($\widetilde{B}$2) and $0 < \alpha(0) \leq 1$, then it satisfies condition (C3).*

Proof. Since $\{\alpha(n)\}$ satisfies condition near-($\widetilde{\mathrm{B}}$2), it is equivalent to a sequence $\{\lambda(n)\}$ of positive real numbers such that $\{\frac{1}{n!\lambda(n)}\}$ is log-concave. Apply Equation (2.1) to the sequence $\beta(n) = \lambda(0)/\lambda(n)$. Then we get

$$\lambda(n)\lambda(m) \leq \lambda(0)2^{n+m}\lambda(n+m) \qquad , \forall n, m \geq 0.$$

Then we can repeat similar arguments as in the proof of Theorem 5.1 to show that the sequence $\{\alpha(n)\}$ satisfies condition (C3). $\square$

For the rest of this section we assume that $u \in C_{+,1/2}$ satisfies conditions (U0) (U2) (U3). We will state several theorems concerning various continuous linear operators acting on $[\mathcal{E}]_u$ and $[\mathcal{E}]_u^*$. These theorems follow from section 3 of the paper [16] as a consequence of the above Theorem 2.6. However, we point out that they can be proved independently without using the corresponding results in the paper [16].

The next theorem corresponds to Theorem 3.1 in [16].

Theorem 5.3. *For any $y \in \mathcal{E}'$, the differential operator D_y is a continuous linear operator from $[\mathcal{E}]_u$ into itself.*

The next theorem corresponds to Theorem 3.2 in [16].

Theorem 5.4. *For any $y \in \mathcal{E}'$, the translation operator T_y is a continuous linear operator from $[\mathcal{E}]_u$ into itself.*

The next theorem corresponds to a fact on page 323 in [16].

Theorem 5.5. *For any $z \in \mathbb{C}$, the scaling operator S_z is a continuous linear operator from $[\mathcal{E}]_u$ into itself.*

For $a, b \in \mathbb{C}$, define the Fourier-Gauss transform $\mathcal{G}_{a,b}\varphi$ of $\varphi \in [\mathcal{E}]_u$ by

$$\mathcal{G}_{a,b}\varphi(x) = \int_{\mathcal{E}'} \varphi(ay + bx)\, d\mu(y). \tag{5.3}$$

Theorem 5.6. *For any $a, b \in \mathbb{C}$, the Fourier-Gauss transform operator $\mathcal{G}_{a,b}$ is a continuous linear operator from $[\mathcal{E}]_u$ into itself.*

For those operators in Theorems 5.3 to 5.6, their adjoints are continuous linear operators from $[\mathcal{E}]_u^*$ into itself. All properties regarding to these operators in the book [21] are all valid with suitable modification. In particular, the integral kernel operators in Chapter 10 and white noise integration in Chapter 13 can be extended to the Gel'fand triple $[\mathcal{E}]_u \subset (L^2) \subset [\mathcal{E}]_u^*$.

Theorem 5.7. *The space $[\mathcal{E}]_u^*$ is closed under the Wick product and the mapping $(\Phi, \Psi) \mapsto \Phi \diamond \Psi$ is jointly continuous from $[\mathcal{E}]_u^* \times [\mathcal{E}]_u^*$ into $[\mathcal{E}]_u^*$ with respect to the inductive limit convex topology.*

Proof. By Fact 2.6, u^* belongs to $C_{+,1/2}$ and is increasing and (log, x^2)-convex. Hence we may apply the Remark of Lemma 4.2 to u^* to see that u^* and $(u^*)^2$ are equivalent. Hence by Theorem 3.2 we can see that the space $S[\mathcal{E}]_u^*$ is closed under multiplication and hence $[\mathcal{E}]_u^*$ is closed under Wick product by definition. Similarly to Theorem 3.5 in [16], we can show the inequality

$$\|\Phi \diamond \Psi\|_{-q,(u)} \leq c\, \|\Phi\|_{-p,(u)} \|\Psi\|_{-p,(u)}, \quad \forall \Phi, \Psi \in [\mathcal{E}_p]_u^*$$

for any p, $c > 1$ and $q > p + \gamma(c)$ with a suitable constant $\gamma(c)$. The joint continuity can be proved applying Lemma A in Appendix to $X' = [\mathcal{E}]_u^*$. $\square$

The next theorem is for the Wick product of test functions. It corresponds to Theorem 3.4 in [16].

Theorem 5.8. *The space $[\mathcal{E}]_u$ is closed under the Wick product and the mapping $(\varphi, \psi) \mapsto \varphi \diamond \psi$ is continuous from $[\mathcal{E}]_u \times [\mathcal{E}]_u$ into $[\mathcal{E}]_u$.*

For the pointwise multiplication of test functions we have the next theorem which corresponds to a fact on page 326 in [16].

Theorem 5.9. *The space $[\mathcal{E}]_u$ is closed under pointwise multiplication and the mapping $(\varphi, \psi) \mapsto \varphi\psi$ is continuous from $[\mathcal{E}]_u \times [\mathcal{E}]_u$ into $[\mathcal{E}]_u$.*

Appendix

To prove the following Lemma A we borrow the idea in parts from the proof of Lemma 2.1 in [19]. Let us give a short remark. It is shown in [19] that the mapping from a dual space $\mathcal{E}^*$ of $\mathcal{E}$ to a n-fold symmetirc tensor product space $(\mathcal{E}^{\otimes n})^*_{symm}$ is continuous with respect to the inductive limit convex topology. The nuclearity of $\mathcal{E}$ plays an essential role to prove this fact. On the other hand, as illustrated in Lemma A, the nuclearity of $\mathcal{E}$ is not an intrinsic assumption to verify the continuity of the mapping from a direct product space $[\mathcal{E}]^*_u \times [\mathcal{E}]^*_u$ to $[\mathcal{E}]^*_u$. That is, we have

Lemma A. *Let X be a complete σ-normed space with norms $\|\cdot\|_1 \le \|\cdot\|_2 \le \cdots \le \|\cdot\|_p \le \cdots$, and let X' be its dual and $\|\cdot\|_{-p}$ the dual norm of $\|\cdot\|_p$. Suppose that X' is an algebra with multiplication $xy \in X'$ of $x, y \in X'$ and that for any $p \ge 1$ there exist an integer $\gamma(p) \ge p$ and a positive constant $C(p)$ such that*

$$\|xy\|_{-\gamma(p)} \le C(p)\|x\|_{-p}\|y\|_{-p}$$

holds for any $x, y \in X'$ with $\|x\|_{-p}, \|y\|_{-p} < \infty$. Then the mapping $(x, y) \mapsto xy$ is jointly continuous from $X' \times X'$ into X' with respect to the inductive limit convex topology.

Proof. Recall that fundamental neighborhoods of x can be given in the form

$$V(x; q, \{\epsilon_p\}_{p\ge q}) = \mathrm{conv}\left(\bigcup_{p\ge q}\{z; \|z\|_{-p} < \epsilon_p\}\right) + x,$$

for a given $q \ge 1$ and a positive sequence $\{\epsilon_p\}_{p\ge q}$ (cf. [19]). Here "conv" means the convex hull, just the collection of all finite convex sums. For $V(0; q, \{\epsilon_p\}_{p\ge q})$, put

$$\delta_p = \min\left\{1, \frac{\epsilon_{\gamma(p)}}{C(p)}\right\} \quad \text{for} \quad p \ge q.$$

For $x, y \in V(0; q, \{\delta_p\}_{p\ge q})$, there exist $\{\alpha_p\}_{p\ge q}^N, \{\beta_p\}_{p\ge q}^{N'}$ and $\{x_p\}_{p\ge q}^N, \{y_p\}_{p\ge q}^{N'}$ such that

$$x = \sum_{p=q}^N \alpha_p x_p, \quad y = \sum_{p=q}^{N'} \beta_p y_p \quad \text{and} \quad \|x_p\|_{-p} < \delta_p, \quad \|y_p\|_{-p} < \delta_p,$$

$$\sum_{p=q}^{N'} \alpha_p = 1, \quad \sum_{p=q}^{N'} \beta_p = 1, \quad \alpha_p \ge 0, \quad \beta_p \ge 0.$$

Then put $\ell_0 = \min\{\gamma(p) : p \geq q\}$ and put

$$z_\ell = \frac{1}{\lambda(\ell)} \sum_{\gamma(k)=\ell} \sum_{p \vee p' = k} \alpha_p \beta_{p'} x_p y_{p'}, \quad \lambda(\ell) = \sum_{\gamma(k)=\ell} \sum_{p \vee p' = k} \alpha_p \beta_{p'}.$$

for $\ell \geq \ell_0$. Now estimate norms of $z_\ell' s$. Since $\|x_p\|_{-k} \leq \|x_p\|_{-p} < \delta_p$ and $\|y_{p'}\|_{-k} \leq \|y_{p'}\|_{-p'} < \delta_{p'}$ for $k \geq p \vee p'$, we have

$$\begin{aligned}
\|z_\ell\|_{-\ell} \leq & \frac{1}{\lambda(\ell)} \sum_{\gamma(k)=\ell} \sum_{p \vee p' = k} \alpha_p \beta_{p'} \|x_p y_{p'}\|_{-\ell} \\
\leq & \frac{1}{\lambda(\ell)} \sum_{\gamma(k)=\ell} \sum_{p \vee p' = k} \alpha_p \beta_{p'} C(k) \|x_p\|_{-k} \|y_{p'}\|_{-k} \\
\leq & \frac{1}{\lambda(\ell)} \sum_{\gamma(k)=\ell} \sum_{p \vee p' = k} \alpha_p \beta_{p'} C(k) \|x_p\|_{-p} \|y_{p'}\|_{-p'} \\
\leq & \frac{1}{\lambda(\ell)} \sum_{\gamma(k)=\ell} \sum_{p \vee p' = k} \alpha_p \beta_{p'} C(k) \delta_k \\
\leq & \epsilon_\ell.
\end{aligned}$$

Since

$$xy = \sum_{\ell \geq \ell_0} \lambda(\ell) z_\ell, \quad \sum_{\ell \geq \ell_0} \lambda(\ell) = 1, \quad \ell_0 \geq q,$$

we obtain $xy \in V(0; q, \{\epsilon\}_{p \geq q})$. Hence the product is jointly continuous at 0.

Next we show the joint continuity at (x_0, y_0). Suppose that $\|x_0\|_{-p_0}, \|y_0\|_{-p_0'} < \infty$. For any given $V(0; q, \{\epsilon_p\}_{p \geq q})$, put $q_0 = \max\{q, p_0, p_0'\}$ and

$$\delta_p = \min\left\{1, \frac{\epsilon_{\gamma(p)}}{3C(p)(1 + \|x_0\|_{-p} + |y_0\|_{-p})}\right\} \quad \text{for} \ \ p \geq q_0$$

and take their neighborhoods as $V(x_0; q_0, \{\delta_p\}_{p \geq q_0})$ and $V(y_0; q_0, \{\delta_p\}_{p \geq q_0})$ and let x and y be in these neighborhoods, respectively. Then we have

$$x = x_0 + \sum_{p=q_0}^{N} \alpha_p x_p, \quad y = y_0 + \sum_{p=q_0}^{N'} \beta_p y_p$$

as above. Then we see

$$xy - x_0 y_0 = (x - x_0)(y - y_0) + (x - x_0) y_0 + (y - y_0) x_0.$$

The first term of the right hand side belongs to $V(0; q, \{\frac{1}{3}\epsilon_p\}_{p \geq q})$. Observe the second term. Put $\ell_0 = \min\{\gamma(k); k \geq q_0\}$ and

$$z_\ell' = \frac{1}{\lambda'(\ell)} \sum_{\gamma(k)=\ell, k \geq q_0} \alpha_k x_k y_0, \quad \lambda'(\ell) = \sum_{\gamma(k)=\ell, k \geq q_0} \alpha_k.$$

Then

$$\begin{aligned}\|z'_\ell\|_{-\ell} \leq& \frac{1}{\lambda'(\ell)} \sum_{\gamma(k)=\ell, k\geq q_0} \alpha_k \|x_k y_0\|_{-\ell} \\ \leq& \frac{1}{\lambda'(\ell)} \sum_{\gamma(k)=\ell, k\geq q_0} \alpha_k C(k) \|x_k\|_{-k} \|y_0\|_{-k} \\ \leq& \frac{1}{\lambda'(\ell)} \sum_{\gamma(k)=\ell, k\geq q_0} \alpha_k C(k) \delta_k \|y_0\|_{-k} < \frac{1}{3}\epsilon_\ell\end{aligned}$$

This implies that $(x - x_0)y_0 \in V(0; q, \{\frac{1}{3}\epsilon_p\}_{p\geq q})$. In the same way, we see $(y - y_0)x_0 \in V(0; q, \{\frac{1}{3}\epsilon_p\}_{p\geq q})$. Thus we have $xy - x_0 y_0 \in V(0; q, \{\epsilon_p\}_{p\geq q})$. We complete the proof. □

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